Chapter 4

An Introduction to

Numerical Methods for Parabolic Equations

First Session Contents:

1) **Introduction**
2) **Finite Difference Methods**
3) **Explicit methods for Parabolic Equations**
4) **Truncation Error (T.E.)**
5) **Consistency**
6) **Stability**

Parabolic PDE

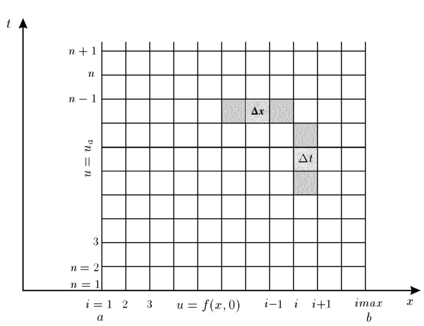
In this chapter, we will discuss about the numerical solution of one-dimensional parabolic PDE as given below:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad a < x < b$$

B.C.'s $\begin{cases} u(a,t) = u_a(t) \\ u(b,t) = u_b(t) \end{cases}$

I.C. $u(x,0) = f(x)$

Computational Domain for Parabolic PDEs



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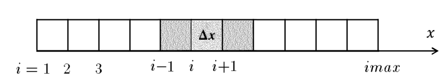
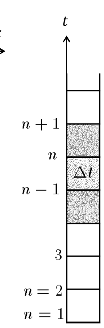




Grid Points or Mesh Points

$x_i = (i-1)\Delta x = (i-1)h \quad i = 1, 2, 3, \ldots imax$

$t_n = (n-1)\Delta t = (n-1)k \quad n = 1, 2, 3, \ldots$

$k = \Delta t$ **Time Step**

$h = \Delta x$ **Grid Spacing**

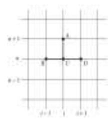
Discretization

General PDE $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Transient Term $\quad \left.\dfrac{\partial u}{\partial t}\right|_i^n = \dfrac{u_i^{n+1} - u_i^n}{k} + O(k)$ (Forward Difference)

Diffusion Term $\quad \left.\dfrac{\partial^2 u}{\partial x^2}\right|_i^n = \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + O(h^2)$ (Central Difference)

Substituting transient and diffusion terms in the PDE, we have



$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + O(k, h^2)$$

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad \text{where} \quad r = \frac{k}{h^2}$$

$$u_i^{n+1} = u_i^n + r\delta_x^2 u_i^n = (1 + r\delta_x^2)u_i^n$$ where δ_x is the central difference operator

Explicit Methods for Parabolic PDEs

$\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ 1-D transient Heat Conduction Problem

Using the following Eq., the temperature at time step n+1 can be obtained from the temperature at time step n.

$$u_i^{n+1} = ru_{i+1}^n + (1-2r)u_i^n + ru_{i-1}^n \quad , \quad r = \frac{k}{h^2} \quad , \quad \text{T.E.} = O(k, h^2)$$

At time step n=1, temperature is known as initial condition.
Therefore,
Temperature at n+1, n+2, … can be obtained.
$[u^{n+1}]=[A][u^n]$

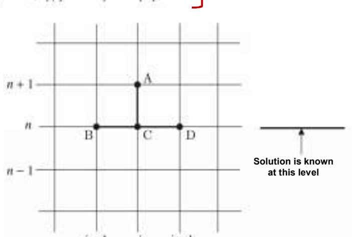
A method which calculates the state of a system at a later time from the state of the system at the current time is called
Explicit Method

Explicit Method (FTCS)

Transient Term (Forward Difference)
Diffusion Term (Central Difference)

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

FTCS
Forward **T**ime **C**entral **S**pace

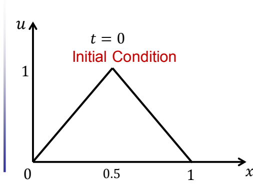


Example (FTCS)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

Initial Condition $\quad u(x,0) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$

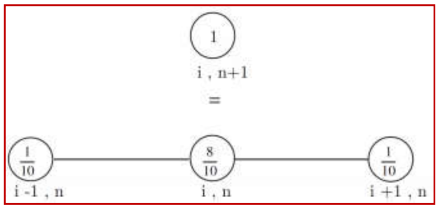
Boundary Conditions $\quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$



$h = \dfrac{1}{10} \qquad k = \dfrac{1}{1000} \qquad r = \dfrac{k}{h^2} = \dfrac{1}{10}$

$$u_i^{n+1} = \frac{1}{10}(u_{i-1}^n + 8u_i^n + u_{i+1}^n)$$

Example (FTCS)



$$\boxed{u_i^{n+1} = \frac{1}{10}(u_{i-1}^n + 8u_i^n + u_{i+1}^n)}$$

Example (FTCS)

Numerical Solution

$$u_i^{n+1} = \frac{1}{10}(u_{i-1}^n + 8u_i^n + u_{i+1}^n)$$

$$u_6^2 = \frac{1}{10}(u_5^1 + 8 \times u_6^1 + u_7^1) = \frac{1}{10}[0.8 + (8 \times 1) + 0.8] = 0.9600$$

$$u_5^3 = \frac{1}{10}(u_4^2 + 8 \times u_5^2 + u_6^2) = \frac{1}{10}[0.6 + (8 \times 0.8) + 0.96] = 0.7960$$

Exact Solution

$$u = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\sin\frac{n\pi}{2})(\sin n\pi x)\exp(-n^2\pi^2 t)$$

Example (FTCS)

$r = \frac{k}{h^2} = 0.10$

Time	$i=1$ $x=0$	$i=2$ $x=0.1$	$i=3$ $x=0.2$	$i=4$ $x=0.3$	$i=5$ $x=0.4$	$i=6$ $x=0.5$	$i=7$ $x=0.6$
0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
0.001	0	0.2000	0.4000	0.6000	0.8000	0.9600	0.8000
0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280	0.7960
0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597	0.7732
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.01	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.02	0	0.1938	0.3781	0.5373	0.6486	0.6891	0.6486

Example (FTCS)

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Symmetry

Example (FTCS)

$x = 0.30$

Time	Numerical Solution ($x = 0.3$)	Exact Solution ($x = 0.3$)	Absolute Error	Relative Error %
$t = 0.005$	0.5971	0.5966	0.0005	0.08
$t = 0.01$	0.5822	0.5799	0.0023	0.4
$t = 0.02$	0.5373	0.5334	0.0039	0.7
$t = 0.1$	0.2472	0.2444	0.0028	1.1

$x = 0.50$

Time	Numerical Solution ($x = 0.5$)	Exact Solution ($x = 0.5$)	Absolute Error	Relative Error %
$t = 0.005$	0.8597	0.8404	0.0193	2.3
$t = 0.01$	0.7867	0.7743	0.0124	1.6
$t = 0.02$	0.6891	0.6809	0.0082	1.2
$t = 0.1$	0.3056	0.3021	0.0035	1.2

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Example (FTCS)

$x = 0.30$

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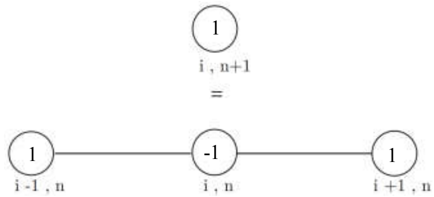
Example (FTCS)



$$h = \frac{1}{10} \qquad k = \frac{1}{100} \qquad r = \frac{k}{h^2} = 1$$

$$u_i^{n+1} = u_{i-1}^n - u_i^n + u_{i+1}^n$$

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Example (FTCS)

$r = 0.10$

t	$i = 1$ $x = 0$	$i = 2$ $x = 0.1$	$i = 3$ $x = 0.2$	$i = 4$ $x = 0.3$	$i = 5$ $x = 0.4$	$i = 6$ $x = 0.5$	$i = 7$ $x = 0.6$
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$r = 1$

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0.00	0	0.2	0.4	0.6	0.8	1.0	0.8
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0.02	0	0.2	0.4	0.6	0.4	1.0	0.4
0.03	0	0.2	0.4	0.2	1.2	−0.2	1.2
0.04	0	0.2	0.0	1.4	−1.2	2.6	−1.2

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Example (FTCS)

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0.03	0	0.2	0.4	0.2	1.2	−0.2	1.2
0.04	0	0.2	0.0	1.4	−1.2	2.6	−1.2

The value of r plays an important role in explicit methods

$$0 < r \le \frac{1}{2}$$

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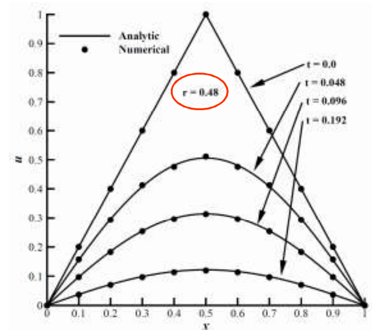
Example (FTCS)



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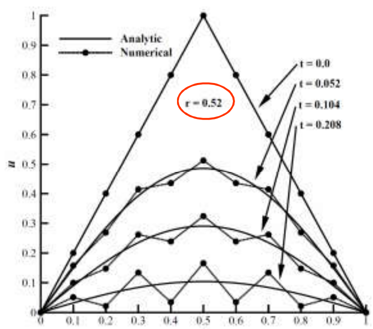
Example (FTCS)



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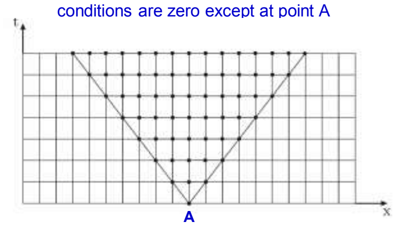
Propagation Speed of a Disturbance

Consider a 1-D parabolic PDE with all initial conditions are zero except at point A



$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

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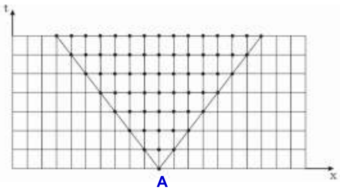
Propagation Speed of a Disturbance



Slope of Characteristics lines:

$$\frac{dt}{dx} = \frac{\Delta t}{\Delta x} \neq 0$$

The propagation speed of a disturbance in parabolic PDEs for given time is infinity

?

The propagation speed of a disturbance in finite-difference form of the Parabolic PDE for given time is $\Delta x / \Delta t$

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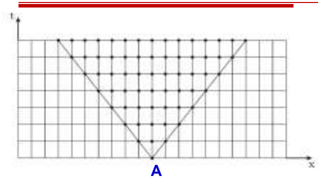
Propagation Speed of a Disturbance



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The propagation speed of a disturbance in parabolic PDEs for given time is infinity

?

The propagation speed of a disturbance in finite-difference form of the Parabolic PDE for given time is $\Delta x / \Delta t$

Solution: $\frac{\Delta x}{\Delta t} = \frac{\Delta x}{r(\Delta x)^2} = \frac{1}{r\Delta x} \quad \left\{ \begin{array}{l} \Delta x \to 0 \\ r = \text{Const.} \end{array} \right\} \Longrightarrow \frac{dt}{dx} \to 0$

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Truncation Error (T.E.)

$L[u] = f \qquad (x,t) \in \Omega$ General PDE

$D[U_i^n] = f_i^n \qquad (x,t) \in \Omega$ Discretized PDE

$U_i^n \approx u_i^n = u(x_i, t_n)$ If h and k are small enough then T.E. approaches to zero and the discretized form is a good approximation of general PDE

Truncation Error is Defined as $\boxed{T_i^n = D[u_i^n] - f_i^n}$ as $h, k \to 0$ independently

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Truncation Error (T.E.)

$$\underbrace{\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} = \underbrace{\frac{u_i^{n+1}-u_i^n}{\Delta t}-\frac{1}{(\Delta x)^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n)}_{\text{FDE}} + \underbrace{\left[-\frac{\partial^2 u}{\partial t^2}\Big|_i^n\frac{\Delta t}{2}+\frac{\partial^4 u}{\partial x^4}\Big|_i^n\frac{(\Delta x)^2}{12}+\dots\right]}_{\text{T.E.}}$$

T.E. is obtained by writing the Taylor series of each term around point (x_i, t_n)

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Truncation Error (T.E.)

$$\underbrace{\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} = \underbrace{\frac{u_i^{n+1}-u_i^n}{\Delta t}-\frac{1}{(\Delta x)^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n)}_{\text{FDE}} + \underbrace{\left[-\frac{\partial^2 u}{\partial t^2}\bigg|_i^n \frac{\Delta t}{2}+\frac{\partial^4 u}{\partial x^4}\bigg|_i^n \frac{(\Delta x)^2}{12}+\ldots\right]}_{\text{T.E.}}$$

T.E. is obtained by writing the Taylor series of each term around point (x_i, t_n)

- Is the finite difference form of PDE acceptable?
- Does the marching method give a good approximation of PDE?

The finite-difference form of PDE should satisfy both **Stability** and **Consistency** conditions

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Consistency

Finite-difference form of a PDE is consistent if:

$$\lim_{h,k\to 0} \text{T.E.}_i^n = 0$$

In some finite-difference methods the T.E. is $\frac{O(\Delta x)}{O(\Delta t)}$

These methods are consistent if $\frac{\Delta x}{\Delta t} \to 0$

DuFort-Frankel method

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} = \frac{1}{(\Delta x)^2}[u_{i+1}^n - \underbrace{(u_i^{n+1}+u_i^{n-1})}_{2u_i^n} + u_{i-1}^n]$$

T.E. $\leftarrow$ $\boxed{+\frac{1}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_i^n (\Delta x)^2 - \frac{\partial^2 u}{\partial t^2}\bigg|_i^n \left(\frac{\Delta t}{\Delta x}\right)^2 - \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\bigg|_i^n (\Delta t)^2}$

The DuFort-Frankel method is consistent if $\lim_{\Delta x,\Delta t\to 0}\left(\frac{\Delta x}{\Delta t}\right) = 0$

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Stability

- Concept of stability occurs in marching problems
- A numerical method is **stable**
  If the errors of any kind (Round of error and Truncation error) will not grow (increasing unconditionally) during time marching.
- Generally, analysis of Consistency in a numerical method is more easier than analysis of Stability
- An explicit method is stable if $r = [\frac{\Delta t}{(\Delta x)^2}] \le \frac{1}{2}$

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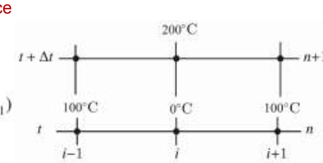
Example

Forward Time Central Space



$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$r = 1$$

$$u_i^{n+1} = u_{i-1}^n - u_i^n + u_{i+1}^n$$

u^{n+1}=100-0+100=200

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Convergence for Marching Problems

A numerical method converges if its global discretization error approaches zero as the mesh is refined

$$\lim_{h,k\to 0} | U_i^n - u_i^n | = 0 \qquad (x_i, t_n) \in \Omega$$

Generally, a numerical method **converges** if it is both **stable** and **consistence**

Lax's Equivalence Theorem

Given a well-posed initial value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence

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